# Final Exam - Numerical Computing
# B. Math I

22 April, 2024

(i) Duration of the exam is 3 hours.

(ii) The maximum number of points you can score in the exam is 100 (total = 110).

(iii) You are not allowed to consult any notes or external sources for the exam.

Name: ______________________________

Roll Number: ______________________________

1. (15 points) Let $A$ be an invertible matrix in $M_{202}(\mathbb{R})$ such that $\|A\| = 100$ and $\|A\|_F = 101$ (where $\|\cdot\|_F$ denotes the Frobenius norm.) With justification, find the sharpest lower bound for the condition number of $A$ (defined as $\kappa(A) = \|A\| \cdot \|A^{-1}\|$.)

   Total for Question 1: 15

2. (15 points) Let $A$ be an $m \times n$ real matrix and $\vec{b} \in \mathbb{R}^m$. Show that there is a unique vector, $\vec{x}$, of smallest norm in $\mathbb{R}^n$ that minimizes $\|A\vec{x} - \vec{b}\|_2$, and that it is given by $\vec{x} = (A^T A)^\dagger A^T \vec{b}$.

   Total for Question 2: 15

3. (15 points) Let $(x_i, f(x_i), f'(x_i)), i = 1, \ldots, n$ be given. Let

$$P_i(x) = a_i + b_i(x - x_i) + c_i(x - x_i)^2 + d_i(x - x_i)^3,$$

be the Hermite cubic interpolant in the range $[x_i, x_{i+1}]$. Show that the following constraints:

$$P_i(x_i) = f(x_i), P_i'(x_i) = f'(x_i), P_i(x_{i+1}) = f(x_{i+1}), P_i'(x_{i+1}) = f'(x_{i+1}), 1 \le i \le n-1,$$

imply that:

$$\begin{aligned} a_i &= f(x_i) \\ b_i &= f'(x_i) \\ c_i &= \frac{3f[x_i, x_{i+1}] - 2f'(x_i) - f'(x_{i+1})}{(x_{i+1} - x_i)} \\ d_i &= \frac{f'(x_i) - 2f[x_i, x_{i+1}] + f'(x_{i+1})}{(x_{i+1} - x_i)^2}. \end{aligned}$$

Total for Question 3: 15

4. Consider a function $f : \mathbb{R} \to \mathbb{R}$, and let $x_0, x_1, \ldots, x_n$ be distinct points in $\mathbb{R}$.
   (a) (10 points) Prove the following formula for the $n$th divided difference,

   $$f[x_0, x_1, \ldots, x_n] = \sum_{i=0}^{n} \frac{f(x_i)}{w'(x_i)},$$

   where $w(x) = (x - x_0)(x - x_1) \cdots (x - x_n)$.
   (b) (10 points) Assuming $f$ is smooth, with justification calculate the limit of the above formula for $f[x_0, x_1, \ldots, x_n]$ as $x_2 \to x_1$, while all other points remain fixed.

Total for Question 4: 20

5. Below is a list of the first 5 Legendre polynomials.

| $n$ | $P_n(x)$ |
|---|---|
| 0 | 1 |
| 1 | $x$ |
| 2 | $\frac{1}{2}(3x^2 - 1)$ |
| 3 | $\frac{1}{5}(5x^3 - 3x)$ |
| 4 | $\frac{1}{8}(35x^4 - 30x^2 + 3)$ |

   (a) (15 points) Compute the weights for the Gaussian quadrature method for 3 nodes and use it to write a formula for a rule of numerical integration to approximate $\int_{-1}^{1} f(x)\,dx$ for a function $f$.
   (b) (10 points) Show that the above rule gives an exact answer for evaluating the integral of polynomials (over the interval $[-1, 1]$) of degree $\le 5$.

Total for Question 5: 25

6. (10 points) In the attached code, there are three errors. State the line numbers which contain the errors and the relevant corrected code.

Total for Question 6: 10

7. (10 points) Manually perform three steps of Euler's method to solve:

$$\frac{dy}{dt} = \frac{1}{t+y+1}, y(0) = 0,$$

with $h = 0.2$.

Total for Question 7: 10

```
function I = simpson(fun,a,b,npanel)
% simpson    Composite Simpson's rule
%
% Synopsis:  I = simpson(fun,a,b,npanel)
%
% Input:     fun    = (string) name of m-file that evaluates f(x)
%            a, b   = lower and upper limits of the integral
%            npanel = number of panels to use in the integration
%
% Output:    I = approximate value of the integral from a to b of f(x)*dx

% Panel is the interval where a rule of integration is to be used (such as Simpson's rule in this case)

n = 2*npanel;        %  total number of nodes
h = (b-a)/n;         %  stepsize
x = a:h:b;           %  divide the interval
f = feval(fun,x);    %  evaluate integrand

I = (h/3)*( f(1) + 3*sum(f(2:2:n-1)) + 3*sum(f(3:2:n-2)) + f(n) );
%           f(a)         f_even              f_odd         f(b)
```